FUNCTIONAL DEPENDENCE*

BY

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1. **Introduction.** The condition for dependence of n functions of $n+p$ variables is roughly that every determinant of order n formed from the matrix of the first partial derivatives vanish identically. The theorem easiest to prove assumes condition (A): *One of the determinants of highest order which do not vanish identically is different from zero at a given point.* The first theorem free from condition (A) is due to Bliss,† who established an analytic relation in the case of two analytic functions of not more than two variables. Osgood‡ proved that, for the case of three or more analytic functions of as many variables, the identical vanishing of the Jacobian does not necessarily imply that the functions satisfy an analytic relation. No result of a positive nature was given in this case.

More recently, Knopp and Schmidt§ have established a relation for the case of n real functions, of class C',|| of not more than n variables. The result is obtained in the large, and free of condition (A).¶

In the present paper we treat first the real case, extending the results

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† G. A. Bliss, *Fundamental Existence Theorems*, Colloquium Publications of the American Mathematical Society, vol. 3, part 1, 1913. See also Osgood, *Lehrbuch der Funktionentheorie*, vol. 2, part 1, chapter 2, §24, where a treatment involving parameters is given. We refer to the latter book as Osgood II.

‡ W. F. Osgood, *On functions of several complex variables*, these Transactions, vol. 17 (1916), pp. 1–8.

§ K. Knopp and R. Schmidt, *Funktionaldeterminanten und Abhängigkeit von Funktionen*, Mathematische Zeitschrift, vol. 25 (1926), pp. 373–381. We refer to this paper, and to the authors, as K and S.

|| A function of class $C^{(k)}$ is one having all partial derivatives, continuous, of order k. A function of class $C^{(\infty)}$ is one having all partial derivatives, continuous, of every finite order.

¶ For a proof of the same results under the weaker hypotheses that the functions need not be of class C' but are merely differentiable in the sense of Stolz, see A. Ostrowski, *Funktionaldeterminanten und Abhängigkeit von Funktionen*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 36 (1927), pp. 129–134.

We mention also a paper by G. Doetsch, *Die Funktionaldeterminante als Deformationsmass einer Abbildung und als Kriterium der Abhängigkeit von Funktionen*, Mathematische Annalen, vol. 99 (1928), pp. 590–601. He defines a point as regular if the matrix of first partial derivatives has the same rank there as at all sufficiently nearby points, otherwise singular. He establishes a functional relation without restriction on the number of variables, under each of the following hypotheses: (i) there are no singular points in the set under consideration; (ii) the singular points are mapped on a set of measure zero; (iii) the singular points lie on an at most denumerably infinite set of n-dimensional manifolds.

of K and S to the case that the number of variables is arbitrary (but finite). The proof of K and S does not generalize in any obvious manner so as to give this result. No conditions like those of Doetsch are imposed. Conditions of differentiability are imposed. It would be interesting to know to what extent, if any, these conditions are necessary. [Added in proof: See footnote to Theorem 4.II.]

In the second part we treat the case of n analytic functions of any finite number of variables. We establish here an important property which would obtain if an analytic relation did exist, namely that the point set determined by the given functions in the representing space for the values of the functions has the properties that it is nowhere dense* and that it cannot disconnect any region† of the space.

Finally we treat the case of two analytic functions of n complex variables, extending the results of Bliss to the case that n is no longer restricted not to exceed 2. Our relation, like that of Bliss, is satisfied only by the values of the given function. In this case we construct a new proof, different from that of Bliss.

In all cases parameters are included. In the case of reals, a relation is obtained which is satisfied only by the values of the functions, which is not the case with K and S.‡

2. **Preliminary lemmas.** We now introduce some lemmas which are helpful in the subsequent proofs.

LEMMA 2.I. *Let a transformation*

$$u_j = v_j(x_1, \cdots, x_n) = v_j(x) \qquad (j = 1, \cdots, m), m > 1, n > 1, \tag{2.1}$$

be given, where $v_j(x)$ is real and continuous over a closed bounded set K of real (x)-space, with B a closed subset of K. Then if each point of B has a neighborhood on B whose image under (2.1) *is nowhere dense in real (u)-space, and if each point of $K-B$ has a neighborhood on $K-B$ whose image is likewise nowhere dense, the image of K in (u)-space is nowhere dense.*§

Since the sum of a finite number of nowhere dense sets is nowhere dense, it follows easily upon use of the Heine-Borel theorem that the image of B is nowhere dense. If Q is any point of (u)-space, and $\mathcal{N}$ any neighborhood of

* This first property also follows easily from the results in the case of reals.

† Region denotes connected open set, hence connected by curves (Hausdorff, *Mengenlehre*, 2d edition, p. 154, Theorem VIII). Domain denotes (non-vacuous) open set.

‡ To obtain this result we modify the treatment in one of the two parts of K and S. It was found necessary to replace the other part of K and S by a different treatment.

The present paper is complete in itself.

§ A set S is dense at a point if the point has a neighborhood which consists of limit points of S.

Q, a set $\mathcal{D}(B)$ can be found open on K and containing B, with image having no point on a sub-neighborhood $\mathcal{N}_1$ of $\mathcal{N}$. By applying the Heine-Borel theorem to $K-\mathcal{D}(B)$, we then find that the image of K has no point on a sub-neighborhood $\mathcal{N}_2$ of $\mathcal{N}_1$. As Q is arbitrary, we infer that the lemma is true.

LEMMA 2.II. *Under the hypotheses of the preceding lemma, suppose each point of B has a neighborhood on B whose image in (u)-space is not only nowhere dense but also has the property that it cannot disconnect any region of (u)-space, and each point of $K-B$ has a neighborhood on $K-B$ whose image satisfies the same conditions. Then the image of K in (u)-space is nowhere dense and cannot disconnect any region of (u)-space.*

As the proof is similar to that of Lemma 2.I, we omit it.

LEMMA 2.III. *Given the equations* (2.1) *with $v_j(x)$ of class C', and with the matrix Δ of first partial derivatives of rank $\leqq r$ neighboring a point $(x^0)=(x_1^0, \cdots, x_n^0)$, with $0<r$, suppose $\partial v_1/\partial x_1 \neq 0$ at (x^0). If we substitute the solution*

$$x_1 = \xi_1(u_1, x_2, \cdots, x_n) \tag{2.2}$$

of the first equation in (2.1), *determined at (x^0), in the remaining equations* (2.1), *obtaining*

$$u_j = \zeta_j(u_1, x_2, \cdots, x_n) \qquad (j = 2, \cdots, m), \tag{2.3}$$

then the matrix of the first partial derivatives of the ζ's with respect to $x_2, \cdots, x_n$ has rank $\leqq r-1$ for $(u_1, x_2, \cdots, x_n)$ near $[v_1(x^0), x_2^0, \cdots, x_n^0]$.

If either $n-1$ or $m-1$ is less than r, the result is obvious. In the contrary case, take any r-rowed determinant of the partial derivatives, as

$$\|\alpha\| = \left\| \frac{\partial(\zeta_2, \cdots, \zeta_{r+1})}{\partial(x_2, \cdots, x_{r+1})} \right\|.$$

With the help of the identity

$$\zeta_j(u_1, x_2, \cdots, x_n) \equiv v_j[\xi_1(u_1, x_2, \cdots, x_n), x_2, \cdots, x_n]$$

and the fact that (2.2) is the solution of the first equation (2.1), the $(j-1)$st row of $\|\alpha\|$ can then be written $(j=2, 3, \cdots, r+1)$:

$$\left\| \left[\frac{\partial v_j}{\partial x_1}\left(-\frac{\partial v_1}{\partial x_2}\Big/\frac{\partial v_1}{\partial x_1}\right) + \frac{\partial v_j}{\partial x_2}\right] \cdots \left[\frac{\partial v_j}{\partial x_1}\left(-\frac{\partial v_1}{\partial x_{r+1}}\Big/\frac{\partial v_1}{\partial x_1}\right) + \frac{\partial v_j}{\partial x_{r+1}}\right] \right\|.$$

Since each bracket is a sum of two terms, α equals a sum of 2^r determinants. But any of these which contain at least two columns of first terms are zero, since those columns are proportional. There remain only $r+1$ of the 2^r determinants, and their sum is seen to be the expansion of

$$\beta = \frac{\partial(v_1, v_2, \cdots, v_{r+1})}{\partial(x_1, x_2, \cdots, x_{r+1})}$$

by minors of the first row, except for a factor $\partial v_1/\partial x_1$ of each term. Since Δ is of rank $\leqq r$, $\beta = 0$. Hence $\alpha = 0$ and the lemma is true.

3. A nowhere dense map. Before stating the next lemma, we introduce the following notation. If m and n are integers, $m \neq 0$, let k_m^n and $t(m, n)$ be the integers defined as follows:

$$n/m \leqq k_m^n < (n/m) + 1; \tag{3.1}$$

$$t(m, n) = k_m^n \text{ if } n \leqq 2m, \tag{3.2}$$

$$t(m, n) = t(m, n-1) + k_m^n - 2 \text{ if } n > 2m. \tag{3.3}$$

LEMMA 3.I. *Given the functions* $v_j(x_1, \cdots, x_n), j = 1, \cdots, m$, *of class* $C^{(t)}$, $t \geqq 1$, *in a domain* $\mathcal{D}$ *of real* (x)-*space, let* L *be a closed bounded subset of* $\mathcal{D}$, *at each point of which all the first partial derivatives of all the* v'*s are zero. Suppose* $t \geqq t(m, n)$. *Then under the transformation*

$$u_j = v_j(x_1, \cdots, x_n) \equiv v_j(x) \qquad (j = 1, \cdots, m), \tag{3.4}$$

the image of L *in* (u)-*space is nowhere dense.*

From (3.2) and (3.3) we see that $k_m^n \leqq t(m, n)$, and since $t \geqq t(m, n)$ it follows that $t \geqq k_m^n$.

Let $k = k_m^n$ and L_1 be the locus of points of L at which all partial derivatives of orders 1 to k inclusive, of all the v's, are zero. Let $b > 0$ be a constant such that the distance from L to the boundary, if any, of $\mathcal{D}$ is $> n^{1/2}b$. Given $\eta > 0$, we choose δ, with $0 < \delta < b$, so that at all points not farther than $n^{1/2}\delta$ from L_1, any kth-order partial derivative of v_j, $j = 1, \cdots, m$, is in absolute value less than η. Next we subdivide (x)-space into n-cubes by planes $x_j = p/2^h$, p any integer, choosing h as a positive integer such that $\epsilon = 1/2^h < \delta$. We consider those closed cubes q of this subdivision each of which contains a point of L_1. If we let P' be the image in (u)-space, under (3.4), of a point P on L_1 in a cube q, it follows from Taylor's theorem with the remainder that the other points of q are transformed into points whose coordinates differ from those of P' by less, in any case, than

$$\frac{n^k \epsilon^k}{k!} \cdot \eta = \frac{\zeta}{2} \epsilon^k \qquad \left(\zeta = \frac{2n^k}{k!} \cdot \eta\right).$$

Therefore the transforms of the points of q lie in a cube of edge $\leqq \zeta \epsilon^k$, hence of volume $\leqq \zeta^m \epsilon^{km}$. Since the volume of the original cube q is ϵ^n, the ratio of the volumes is $\zeta^m \epsilon^{km-n}$. Now $km - n \geqq 0$, by (3.1), since $k = k_m^n$; and ζ and ϵ

can be made as small as we like. Since the total volume of the cubes q does not increase as $\epsilon = 1/2^h$ becomes smaller, it follows that L_1 can be enclosed in a finite set of closed cubes whose image has Jordan measure less than a fixed preassigned constant. Hence the image of L_1 has Jordan measure zero, and since the image is a closed set, it must be nowhere dense.*

If $n \leqq m$ then $k=1$ and $L_1 = L$, and the proof of Lemma 3.I is completed. If $n > m$ then $k > 1$ and we continue.

From Lemma 2.I we see that we can now confine our attention to a neighborhood of an arbitrary point of $L - L_1$. First we take the case of a point where all partial derivatives of orders 1 up to $k-1$, of all the v's, are zero. As we are going to treat separately the case (which can arise only if $k > 2$) of a point at which at least one derivative of order $k-1$ is different from zero, it follows, again from Lemma 2.I, that for the present we need merely consider a closed set, say L_2, in a neighborhood of a point Q of $L - L_1$, with all partial derivatives of orders up to $k-1$ vanishing at each point of L_2.

Since Q is not on L_1, at least one derivative of order k is not zero at Q, say $\partial^k v_1/\partial x_1^k \neq 0$. Since $n > m$, $n > 1$. Now we apply the implicit function theorem to the locus $\partial^{k-1} v_1/\partial x_1^{k-1} = 0$, which contains L_2, obtaining that locus in the form

$$x_1 = \xi_1(x_2, \cdots, x_n), \quad \text{of class } C^{(t-k+1)}.\dagger \tag{3.5}$$

We take L_2 small enough so that (3.5) contains L_2. Let l_2 be the locus of points of $(x_2, \cdots, x_n)$-space which, under (3.5), give points of L_2. Now we substitute in (3.4) obtaining the equations

$$u_j = v_j[\xi_1(x_2, \cdots, x_n), x_2, \cdots, x_n] \qquad (j = 1, \cdots, m). \tag{3.6}$$

Let $b_1 > 0$ be a constant such that for points of $(x_2, \cdots, x_n)$-space within distance $(n-1)^{1/2} b_1$ of l_2, (3.5) still holds and gives points within distance $n^{1/2} b$ of L. Let M be the larger of 1 and $(n-1) \cdot$(maximum of $|\partial \xi_1/\partial x_2|, \cdots, |\partial \xi_1/\partial x_n|$ for points within distance $(n-1)^{1/2} b_1$ of l_2). Let $\epsilon_1 > 0$ be a constant, with $0 < \epsilon_1 < b_1$. Then it follows from the law of the mean that if the coordinates of a point $(x_2, \cdots, x_n)$ differ from those of a point of l_2 by not more than ϵ_1, the corresponding values of x_1 differ by at most $M\epsilon_1$. Let H be an upper bound for the absolute values of the partial derivatives of order k, of all the v's, for points of $\mathcal{D}$ within distance $n^{1/2} b$ of L. Now we apply Taylor's theorem with the remainder to the functions in (3.4), for two points of the locus (3.5),

* In using this property to show that the closed set is nowhere dense, namely that it has Jordan measure zero, we follow K and S.

† That the class of ξ_1 is at least that of $\partial^{k-1} v_1/\partial x_1^{k-1}$ is seen easily from the formulas for the derivatives of ξ_1.

one on L_2, whose respective coordinates $x_2, \cdots, x_n$ differ by at most ϵ_1. Hence their coordinates x_1 differ by at most $M\epsilon_1 \geqq \epsilon_1$. In applying Taylor's theorem, all derivatives of orders less than k are taken at the first point, and the remainder term is a sum of derivatives of order k. Therefore, by (3.6), the corresponding two points in (u)-space have coordinates respectively differing by less than

$$\frac{n^k(M\epsilon_1)^k}{k!}\cdot H = \frac{H_1}{2}(\epsilon_1)^k.$$

Therefore if we cover a portion of $(x_2, \cdots, x_n)$-space containing l_2 by a network of $(n-1)$-cubes, of edge ϵ_1, each of which has at least one point in l_2, each of these cubes is mapped by (3.5) and (3.4) on a subset of a cube in (u)-space of edge $H_1(\epsilon_1)^k$. The ratio of the volumes of two such cubes is $(H_1)^m(\epsilon_1)^{km-n+1}$. Since $k=k_m^n$, $km-n+1>0$. As ϵ_1 can be made as small as we like, we conclude that the image of L_2 has Jordan measure zero, hence is nowhere dense.

If $k=2$, Lemma 3.I is now proved. If $n\leqq 2m$ then k must equal 1 or 2, and we see that we have now proved the following:

LEMMA 3.II. *If $n\leqq 2m$, Lemma 3.I is true.*

If $k>2$, we continue with the proof of Lemma 3.I. Again by use of Lemma 2.I we find that we can next turn our attention to a closed subset L_3 of L, neighboring a point of L, with all partial derivatives of $v_1, \cdots, v_m$, of orders 1 to $k-2$ inclusive, zero at each point of L_3, but with some particular $(k-1)$st-order partial derivative, say $\partial^{k-1}v_1/\partial x_1^{k-1}$, not zero in L_3. Applying the implicit function theorem to the locus $\partial^{k-2}v_1/\partial x_1^{k-2}=0$, we obtain $x_1=\xi_2(x_2, \cdots, x_n)$, ξ_2 of class $C^{(t-k+2)}$. We substitute this function into the equations (3.4), obtaining

$$u_j = \zeta_j(x_2, \cdots, x_n) \qquad (j = 1, \cdots, m), \tag{3.7}$$

with ζ_j of class $C^{(t-k+2)}$. We can now apply mathematical induction, since if we reduce the situation to that of m functions of a number of variables not greater than $m/2$, the corresponding k will be not greater than 2, and we can then apply Lemma 3.II. Since in (3.7) we have m functions of $n-1$ variables, of class $C^{(t-k+2)}$, it follows that we can apply the inductive process provided $t-k+2\geqq t(m, n-1)$. Since $t\geqq t(m, n)$ and $k=k_m^n$, (3.3) is seen to ensure the fulfillment of this condition when $n>2m$.

Continuing in this way we finally find that Lemma 3.I is valid provided $t-k+\nu\geqq t(m, n-1)$, for a finite (possibly vacuous) set of values of $\nu>2$. As this inequality is a consequence of the one above, we conclude that Lemma 3.I is true.

Let (cf. (3.1))

$$s(m, n, r) = \overset{n-r}{k}_{m-r} + \overset{n-2m+r-1}{k}_{m-r} + \overset{n-2m+r-2}{k}_{m-r} + \cdots + \overset{1}{k}_{m-r} \quad \text{if } n - 2m + r > 1, \tag{3.8}$$

$$s(m, n, r) = \overset{n-r}{k}_{m-r} \quad \text{if } n - 2m + r \leqq 1. \tag{3.9}$$

THEOREM 3.III. *Given the functions* $v_j(x_1, \cdots, x_n)$, *of class* $C^{(s)}$, $s \geqq 1$ $(j=1, \cdots, m)$, *in a domain* $\mathfrak{D}$ *of real* (x)*-space, let* J *denote the matrix of first partial derivatives. Let* B *be a closed bounded subset of* $\mathfrak{D}$, *at each point of which* J *has rank* $\leqq r < m$, *with* r *a non-negative integer. Suppose* $s \geqq s(m, n, r)$. *Then under the transformation*

$$u_j = v_j(x_1, \cdots, x_n) \qquad (j = 1, \cdots, m), \tag{3.10}$$

the image of B *in* (u)*-space is nowhere dense.**

According to Lemma 3.I, if L is the subset of B at each point of which all the first partial derivatives of all the u's are zero, the image of L is nowhere dense provided

$$t(m, n) \leqq s. \tag{3.11}$$

For the moment we defer the verification of (3.11), and proceed with the rest of the argument.

If L is all of B, nothing remains but to verify (3.11). If L is not all of B, then r must be positive and we proceed as follows. According to Lemma 2.I we need now consider only a closed subset Λ of B consisting of points in a neighborhood of some point Q of $B-L$. Since Q is not on L, some first partial derivative of a v_j is different from zero at Q, say $\partial v_1/\partial x_1 \neq 0$. Applying the implicit function theorem, we solve the first equation (3.10) in the form

$$\begin{aligned} x_1 &= \xi_3(u_1, x_2, \cdots, x_n), \quad \text{if } n > 1, \\ x_1 &= \xi_3(u_1), \qquad\qquad\quad\; \text{if } n = 1, \end{aligned} \tag{3.12}$$

with ξ_3 of class $C^{(s)}$, and substituting the result in the remaining equations (3.10) obtain

$$\begin{aligned} u_j &= \phi_j(u_1, x_2, \cdots, x_n) \qquad (j = 2, \cdots, m), \text{ if } n > 1, \\ u_j &= \phi_j(u_1) \qquad\qquad\qquad\; (j = 2, \cdots, m), \text{ if } n = 1, \end{aligned} \tag{3.13}$$

with ϕ_j of class $C^{(s)}$. (Note that $m>1$ since $r>0$ and $m>r$.) We treat the case $n>1$, as it is then obvious how to treat the case $n=1$. We take Λ so small that it is part of the locus for which (3.12) is valid. We now let λ be

* If $n \leqq m$, the only condition is that the u's be of class C', which is the result of K and S.

the projection of Λ on $(x_2, \cdots, x_n)$-space, and see, by Lemma 2.I, that it will be sufficient to show that the set in $(u_1, \cdots, u_n)$-space obtained from (3.13) by taking $(x_2, \cdots, x_n)$ anywhere in λ and u_1 anywhere in a closed neighborhood of the value determined by (3.10) at Q, is nowhere dense. It is sufficient to prove that the set in $(u_2, \cdots, u_n)$-space obtained from (3.13) for each fixed value of u_1 is nowhere dense, for if a closed set is dense at a point in (u)-space it must contain a neighborhood of the point. According to Lemma 2.III, the rank of the matrix of first partial derivatives of the ϕ's in (3.13) with respect to $x_2, \cdots, x_n$ is $\leqq r-1$. As the theorem is, by Lemma 3.I, proved for $r=0$ (except for verifying (3.11)), we can use induction with respect to the rank, and since in (3.13) with $u_1=$ constant we have $m-1$ functions of $n-1$ variables, it follows that to complete the proof of Theorem 3.III we need merely show, in addition to (3.11), that

$$(3.14) \qquad s(m, n, r) \leqq s(m-1, n-1, r-1) \quad \text{if } r > 0.$$

But if m, n, r are each reduced by unity, $n-r$, $m-r$ and $n-2m+r$ are all unchanged. Hence, by (3.8) and (3.9), (3.14) holds with the equality sign when $r>0$. Therefore it remains only to verify (3.11).

First we note that if $n=2m+1$, then from (3.2) and (3.3) we have $t(m, n)=2+k_m^n-2=k_m^n=3$. Hence

$$(3.15) \qquad t(m, n) = k_m^n \quad \text{if } n \leqq 2m+1.$$

Next, with the help of (3.1), we rewrite (3.3) in the following form, with n replaced by ν:

$$(3.16) \qquad t(m, \nu) = k_m^{\nu-2m} + t(m, \nu-1), \quad \text{if } \nu > 2m.$$

Writing (3.16) for $\nu=n$, then substituting from (3.16) in the result with $\nu=n-1$, etc., we obtain the following, when $n>2m+1$:

$$(3.17) \qquad t(m, n) = k_m^{n-2m} + k_m^{n-2m-1} + \cdots + k_m^1 + t(m, 2m).$$

From (3.2), $t(m, 2m)=2$, and combining the 2 with the first term of the right hand member of (3.17) we obtain

$$(3.18) \quad t(m, n) = k_m^n + k_m^{n-2m-1} + k_m^{n-2m-2} + \cdots + k_m^1, \quad \text{if } n > 2m+1.$$

Case I. $n\geqq 2m+2$. Then $n-2m+r>1$ and (3.8) holds. Since in this case $n>m$, $k_{m-r}^{n-r}\geqq k_m^n$, a relation between the first terms of the sums in (3.8) and (3.18) respectively. The second term in (3.8) is obviously (from (3.1)) at least as great as the second term in (3.18), etc. As there are at least as many terms in (3.8) as in (3.18), it follows that in Case I (3.11) is satisfied, since $s \geqq s(m, n, r)$.

Case II. $m < n \leqq 2m+1$. As in Case I, $k_{m-r}^{n-r} \geqq k_m^n$. It then follows from (3.15) that (3.11) is satisfied, whether (3.8) or (3.9) is used.

Case III. $n \leqq m$. According to (3.15), $t(m, n)=1$, hence again (3.11) is satisfied. This completes the proof of Theorem 3.III.

We see from (3.8), (3.9), (3.15), and (3.18) that $s(m, n, 0)=t(m, n)$. Since, as we have observed, (3.14) is satisfied with the equality sign, and the values of $t(m, n)$ were derived from (3.2) and (3.3) which were used in the proof of Lemma 3.I, it follows that our treatment does not admit any smaller value for s than $s(m, n, r)$ as given by (3.8) and (3.9).

4. Dependence of real functions. After a preliminary theorem, which is an extension of a theorem of K and S, we obtain our theorem on dependence of real functions, following the procedure of K and S.

THEOREM 4.I. *If M is a closed set of the real number space of $w_1, \cdots, w_n$, there exists a function $F(w_1, \cdots, w_n) \equiv F(w)$ of class $C^{(\infty)}$ in all of finite (w)-space, which vanishes only at the points of M in (w)-space.*

First divide (w)-space by the hyperplanes $w_k=0, \pm 1, \pm 2, \cdots$ $(k=1, \cdots, n)$, and let $[q_1]$ denote the set of the resulting closed hypercubes which contain no points of M. For $s=2, 3, \cdots$, we consider the hyperplanes $w_k=0, \pm 1/2^s, \pm 2/2^s, \pm 3/2^s, \cdots$ $(k=1, \cdots, n)$, and, for each s, let $[q_s]$ denote the set of those of the closed hypercubes into which these planes divide (w)-space, which contain no point of M and no inner point of any cube q_j with $j < s$. Next, for $s=1, 2, 3, \cdots$, we define

$$(4.1) \qquad f_s(w) = s^{-s} \cdot \exp \lfloor - (\sin (2^s \pi w_1) \sin (2^s \pi w_2) \cdots \sin (2^s \pi w_n))^{-2} \rfloor$$

in each cube of $[q_s]$, and $f_s(w)=0$ elsewhere. Then we define $f(w)=f_1(w)+f_2(w)+f_3(w)+\cdots$. It is easily verified that $f(w)$ is of class $C^{(\infty)}$. Evidently $f(w)=0$ at each point of M, but $f(w) \not\equiv 0$ in any region.*

Now let $a_1=0$ and a_t denote the positive square root of the $(t-1)$st positive integer which is not a perfect square, $t=2, 3, \cdots, n+1$. Thus $a_2^2=2$, $a_3^2=3$, $a_4^2=5$, etc. For $t=1, 2, \cdots, n+1$, let M_t denote the set obtained from M by subjecting it to the transformation $w_j'=w_j+a_j$ $(j=1, \cdots, n)$, and $\phi_t(w)$ the function defined exactly as $f(w)$ was defined, but with M replaced by M_t in determining $\phi_t(w)$. We denote $\phi_t(w+a_t)=\phi_t(w_1+a_t, w_2+a_t, \cdots, w_n+a_t)$. Let $F(w)=\phi_1(w+a_1)+\phi_2(w+a_2)+\cdots+\phi_{n+1}(w+a_{n+1})$. Then $F(w)$ is of class $C^{(\infty)}$ in (w)-space, since $f(w)$ is, and obviously vanishes

* This paragraph is taken from K and S, where further details are given. We may replace the function in (4.1) by a simpler one, as they did, if $f(w)$ is not required to be of class $C^{(\infty)}$.

on M. Furthermore, $F(w)$ vanishes only on M. For, since $f(w)\geqq 0$, if $F(b)=0$, then $\phi_1(b+a_1)=\phi_2(b+a_2)=\cdots=\phi_{n+1}(b+a_{n+1})=0$. From the definition of $\phi_t(w)$ it follows that for $t=1, 2, \cdots, n+1$ at least one of the sums b_1+a_t, $b_2+a_t, \cdots, b_n+a_t$ would be rational if (b) were not on M. Let $b_{m_t}+a_t$ be one such, $t=1, \cdots, n+1$. Since there are only n b's, we see that one of them must occur twice in such a sum. Thus b_m+a_p and b_m+a_q must both be rational, $p<q$, for some m, p, q. Thus a_p would equal a_q+r, r rational and $\neq 0$ since $a_p\neq a_q$; and by equating the squares of these two expressions, we would have a_q rational. But if a positive integer a_q is not a perfect square its square root is irrational, as follows from the theorem of unique factorization of positive integers. Hence we would have a contradiction, and it follows that Theorem 4.I is true.*

Next we state the definition of functional dependence, including the case that parameters are involved.

Definition 1. *Functions* $v_j(x_1, \cdots, x_n, y_1, \cdots, y_p)$, $j=1, \cdots, m$, *defined on a closed bounded set B of* (x, y)-*space, are said to be dependent in* $x_1, \cdots, x_n$ *on B if there is a function* $F(u_1, \cdots, u_m, y_1, \cdots, y_p)$ *with the following properties.*

(i) $F(u, y)$ *is defined in all of real* (u, y)-*space and has continuous partial derivatives of the first order there.*

(ii) *For each* (y^0), $F(u, y^0)\not\equiv 0$ *in each region of* (u)-*space.*

(iii) $F[v_1(x, y), \cdots, v_m(x, y), y_1, \cdots, y_p]\equiv\Phi(x, y)=0$ *at each point of B.*

Definition 2. *Functions* $v_j(x_1, \cdots, x_n, y_1, \cdots, y_p)$, $j=1, \cdots, m$, *defined in a domain* $\mathfrak{D}$ *of* (x, y)-*space, are said to be dependent in* $x_1, \cdots, x_n$ *on* $\mathfrak{D}$ *if they are dependent in* $x_1, \cdots, x_n$ *on every closed subset of* $\mathfrak{D}$.

Theorem 4.II. *Let* $v_j(x_1, \cdots, x_n, y_1, \cdots, y_p)$ *be given of class* $C^{(s)}$, $s\geqq 1$, *in a domain* $\mathfrak{D}$ *of real* (x, y)-*space*, $j=1, \cdots, m$, *and where the y's may be lacking. Let K be a closed subset of* $\mathfrak{D}$ *at each point of which the matrix of the first partial derivatives of the v's with respect to the x's is of rank* $\leqq r$, *where* $0\leqq r<m$, *r a fixed integer. Suppose* $s\geqq s(m, n, r)$ [*see* (3.1), (3.8), (3.9)]. *Then the functions* $v_j(x, y)$ *are dependent in* $x_1, \cdots, x_n$ *on K.*†

Let M denote the locus of all points of (u, y)-space satisfying the condition that there be at least one set $\xi_1, \cdots, \xi_n$ such that (ξ, y) is on K and $u_j=v_j(\xi, y)$, $j=1, \cdots, m$. Since K is closed and bounded, M is closed (and bounded). Hence, by Theorem 4.I, it is seen that Theorem 4.II is true if M

* A more direct proof without use of the irrational quantities is easily given, but the exposition would be cumbersome.

† The theorem of K and S requires that $n\leqq m$. Added in proof: E. Kamke has proved this theorem for the special case $n=m+1$; see Mathematische Zeitschrift, vol. 39 (1935), pp. 672–676.

is nowhere dense. Since M is closed, it is sufficient to prove that for each fixed set $(y_1, \cdots, y_p)$, the corresponding subset of M is nowhere dense. But this follows from Theorem 3.III. Hence Theorem 4.II is true.

THEOREM 4.III. *Theorem* 4.II *remains true if* $K = \mathfrak{D}$ (*hence not closed*).

This follows from Definition 2.

THEOREM 4.IV. *Theorems* 4.II *and* 4.III *remain true if, in Definition* 1, F *is required to be of class* $C^{(\infty)}$, *and* $F(u, y) = 0$ *only at points for which* $u_j = v_j(x, y)$ *for some* (x) *with* (x, y) *on* B.*

Theorem 4.IV was actually proved in establishing Theorems 4.II and 4.III.

5. **Several functions of several complex variables.** When for $m > 2$ analytic functions of several complex variables the rank of the matrix of first partial derivatives is less than m, it follows from Osgood's examples that we cannot establish the existence of an analytic relation, even in the small. However, we prove a geometric result, which applies in the large.

LEMMA 5.I. *Let the functions* $f_j(x_1, \cdots, x_n, y_1, \cdots, y_p) \equiv f_j(x, y)$, $j = 1, \cdots, m$, *be analytic in a domain* $\mathfrak{D}$ *of the real* $(2n+2p)$-*space of the* $n+p$ *complex variables* $x_1, \cdots, y_p$. *Let* B *be a closed subset of* $\mathfrak{D}$ *at each point of which* $\partial f_j/\partial x_r = 0$, $j = 1, \cdots, m$, $r = 1, \cdots, n$. *Then the points* $(u_1, \cdots, u_m, y_1, \cdots, y_p)$ *of the set* B' *obtained by use of the equations*

$$u_j = f_j(x, y) \qquad (j = 1, \cdots, m), \tag{5.1}$$

at all points of B, *form a set nowhere dense in the* $(2m+2p)$-*dimensional* (u, y)-*space, and having the property that it cannot disconnect any region of that space.*

First we prove that B' is nowhere dense. Let B_0 be the part of B in any subspace defined by $y_k = y_k^0$, $k = 1, \cdots, p$, and B_0' the corresponding part of B'. If there are no y's, we take $B_0 = B$, and $B_0' = B'$. Then we have the functions $u_j = f_j(x, y^0)$ of the x's only, satisfying $\partial f_j/\partial x_r = 0$. Now neighboring any point of B_0 the simultaneous solution of the latter mn equations is given, according to the Second Weierstrass Preparation Theorem,† by a finite number of configurations, each with a certain positive number of independent variables, provided not all the left hand members of the equations are identically zero. From the conditions $\partial f_j/\partial x_r = 0$ holding on each of these configurations it follows that on each of them $f_j = u_j = \text{constant}$, $j = 1, \cdots, m$. In the case that all the left members of the equations are identically zero,

* The last part of this theorem is not proved by K and S in their case $n \leq m$.

† Osgood II, chapter 2, §17.

each f_j is obviously constant. Since B_0 is closed we can apply the Heine-Borel theorem to it, and it follows that

(B_0'): *B_0' contains only a finite number of points.*

Since B' is closed it must then be nowhere dense, for a closed dense set must contain a region.

To prove the second part, we must show that if $\mathfrak{R}$ is any region (connected open set) of (u, y)-space, and $\mathfrak{R}_1=\mathfrak{R}-\mathfrak{R}B'$, then $\mathfrak{R}_1$ is a region. It is obviously open. Now if there are no y's, the result follows from (B_0'). Hence we may suppose that there are some y's. Let C and D be any two points of $\mathfrak{R}_1$, and let them be joined by a path l consisting of straight line segments CS_1, $S_1S_2, \cdots, S_{t-1}S_t, S_tD$, all in $\mathfrak{R}$. We take the segments (by inserting additional points if necessary) in length $<d$, where the distance from l to any boundary point of $\mathfrak{R}$ is greater than $3d$. From property (B_0') it follows that the S's can be changed slightly, in each case keeping the y's fixed, so that none of them is on B'. Let this be done, with everything mentioned above still holding.

We now show how to replace the line segments by paths in $\mathfrak{R}_1$, if they are not already entirely in $\mathfrak{R}_1$. We take CS_1, say. Let C_y, S_{1y} and L denote the projections on (y)-space of C, S_1 and the segment CS_1, respectively. For each point of L we now determine a subsegment of L as follows. To a point A of L, not C_y or S_{1y}, we first let correspond a point $P(A)$ of $\mathfrak{R}_1$ which projects onto (y)-space in the point A. The existence of such a point $P(A)$ follows from property (B_0'). The subsegment of L determined by A is now chosen as one with mid-point at A and so short that the parallel segment through $P(A)$ in (u, y)-space, of the same length and with mid-point at $P(A)$, is within $\mathfrak{R}_1$ and also within a sphere Σ with radius $2d$ and center at the mid-point of CS_1. If A is at C_y or S_{1y}, the corresponding subsegment of L is similarly defined, but has A as one end point. We now apply the Heine-Borel theorem and choose a finite set of these subsegments of L which cover L, and shorten some of them if necessary so that they just cover L but no two of them have more than one point in common. If $C_y, E_1, E_2, \cdots, E_s, S_{1y}$ are the points A determining these subsegments, the corresponding parallel line segments through the points $P(C_y)=C, P(E_1), \cdots, P(E_s), P(S_{1y})=S_1$ will constitute part of the path in $\mathfrak{R}_1$ joining C to S_1. The remainder of that path consists of a finite number of curves constructed as follows.

Let F_1 and F_2 denote end points of two of these line segments which project onto a single point of L (common end point of two of the subsegments of L). It is then sufficient to join F_1 and F_2 by a curve in $\mathfrak{R}_1$. But F_1 and F_2 lie in a space $\mathfrak{S}$ defined by $y_k=\text{constant}$, $k=1, \cdots, p$, which, according to

(B_0'), contains only a finite number of points of B'. Hence any curve interior to Σ and on $\mathcal{S}$, joining F_1 to F_2 and avoiding this finite set of points, will be satisfactory. It follows that Lemma 5.I is true.

THEOREM 5.II. *Let the functions* $f_j(x_1, \cdots, x_n, y_1, \cdots, y_p) \equiv f_j(x, y)$, $j=1, \cdots, m$, *be analytic in a domain* $\mathcal{D}$ *of the real* $(2n+2p)$-*space of the* $n+p$ *complex variables* $x_1, \cdots, y_p$. *Let* K *be a closed subset of* $\mathcal{D}$ *at each point of which the matrix of the first partial derivatives of the f's with respect to the x's is of rank* $\leqq r<n$. *Then the points* $(u_1, \cdots, u_m, y_1, \cdots, y_p)$ *of the set* K' *obtained by use of* (5.1) *at all points of* K, *form a set nowhere dense in the* $(2m+2p)$-*dimensional* (u, y)-*space, and having the property that it cannot disconnect any region of that space.*

The theorem is also true if there are no y's.

Let B denote the (closed) subset of K at each point of which every derivative $\partial f_i/\partial x_j=0$. We now apply Lemma 2.II,as follows. The x's of Lemmas 2.I and 2.II are the real and imaginary parts of the x's and y's of Theorem 5.II. For equations (2.1) we now have the equations resulting from (5.1) involving those parts and the real and imaginary parts of the u's, together with the $2p$ equations resulting from

$$u_j = y_{j-m} \qquad (j = m + 1, \cdots, p). \tag{5.2}$$

Thus the m of Lemma 2.II is $2m+2p$ of Theorem 5.II. Now according to Lemma 5.I, the image B' of B under (5.1) and (5.2) satisfies the conclusion of Theorem 5.II, hence satisfies the first hypothesis of Lemma 2.II. To complete the proof of the theorem we now see that it will be sufficient to show that the second hypothesis of Lemma 2.II is also satisfied.

If $r=0$, $K-B$ is vacuous and no further proof is necessary.

If $K-B$ is not vacuous, at any point of $K-B$ at least one first partial derivative is not zero, say $\partial f_1/\partial x_1 \neq 0$. By the implicit function theorem the first equation of (5.1) then has a solution which can be substituted in the remaining equations (5.1), yielding

$$u_j = g_j(x_2, \cdots, x_n, y_1, \cdots, y_p, u_1) \qquad (j = 2, \cdots, m), \tag{5.3}$$

g_j analytic, with u_1 now one of the parameters, say $u_1=y_{p+1}$. According to Lemma 2.III the matrix of the first partial derivatives of the g's with respect to $x_2, \cdots, x_n$ is of rank $\leqq r-1$. Since Theorem 5.II is true when $r=0$ it is true when $m=1$. Hence if $m>1$ we can apply mathematical induction and assume that it is true for $m-1$ functions. Since $r-1$ is less than $m-1$, it then follows that Theorem 5.II is true for the $m-1$ functions of $n-1$ variables and $p+1$ parameters given by (5.3), here considered only neighboring a point. Hence the second hypothesis of Lemma 2.II is satisfied, and Theorem 5.II is true.

6. **The case of two analytic functions.** We prove the following theorem.*

THEOREM 6.I. *Let* $f(x, y) \equiv f(x_1, \cdots, x_n, y_1, \cdots, y_p)$ *and* $g(x, y)$ *be given, analytic, with the matrix of the first partial derivatives of f and g with respect to* $x_1, \cdots, x_n$ *of rank* <2, *neighboring a point* $P: (x^0, y^0)$ *of the space of the complex variables* $x_1, \cdots, y_p$, *and with* $f(x, y^0)$ *not identically constant. Then f and g satisfy an analytic relation.*

More precisely, let $u_0 = f(x^0, y^0)$ *and* $v_0 = g(x^0, y^0)$. *Then there exists a function* $G(y_1, \cdots, y_p, u, v) \equiv G(y, u, v)$, *a polynomial in v with coefficients analytic in* (y, u) *near* (y^0, u_0), *such that* $G[y, f(x, y), g(x, y)] \equiv 0$; *and if* $G(y^1, u_1, v_1) = 0$ *for* (y^1, u_1, v_1) *in a neighborhood of* (y^0, u_0, v_0), *then* $u_1 = f(x, y^1)$, $v_1 = g(x, y^1)$, *for a* $(2n-2)$*-dimensional set of points* (x) *near* (x^0).†

The theorem is also true if there are no y's.

By a change of variables if necessary we may assume that $0 = u_0 = v_0 = x_1^0 = \cdots = y_p^0$. We also make a change in the x's only so that $f(x_1, 0, \cdots, 0; 0, \cdots, 0) \not\equiv 0$. That this is possible follows from one of the hypotheses. We now establish

$$\phi(x, y, u) \equiv f(x, y) - u \equiv H(x, y, u) \cdot \Omega(x, y, u), \tag{6.1}$$

where $\Omega(x, y, u)$ is analytic and not zero near $(0, 0, 0)$, and

$$\begin{aligned} H(x, y, u) \equiv x_1^m &+ A_1(x_2, \cdots, x_n, y, u) \cdot x_1^{m-1} + \cdots \\ &+ A_m(x_2, \cdots, x_n, y, u) \end{aligned} \tag{6.2}$$

where the A's are analytic near $(0, 0, 0)$, and $A_j(0, 0, 0) = 0$, $j = 1, \cdots, m$. In (6.1) the first identity simply defines ϕ. Next we note that since $f(x_1, 0, \cdots, 0) \not\equiv 0$, $\phi(x_1, 0, \cdots, 0, 0) \not\equiv 0$, so that we can apply the Weierstress Preparation Theorem‡ to ϕ, giving us the second identity in (6.1), together with (6.2).

Since $\phi_u = -1 \neq 0$, ϕ is irreducible§ at $(0, 0, 0)$, hence we see from (6.1) that H is irreducible at $(0, 0, 0)$. Therefore the discriminant‖ $\Delta(x_2, \cdots, x_n,$

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* The theorem of Bliss requires that $n \leq 2$. Cf. Bliss or Osgood, loc. cit.

† A 0-dimensional set is non-vacuous.

‡ Osgood II, chapter 2, §2.

§ An analytic function is reducible at a point P if, neighboring P, it is expressible as the product of two analytic functions, each vanishing at P.

‖ Any definition of discriminant may be used, since if one of them vanishes identically, so do the others.

$y, u)$ of H is not identically zero, and where $\Delta \neq 0$ the roots of $H=0$ are distinct analytic functions.* Let the roots be

$$x_1 = \xi_j(x_2, \cdots, x_n, y, u) \qquad (j = 1, \cdots, m) \tag{6.3}$$

and define

$$\begin{aligned} F(x_2, \cdots, x_n, y, u, v) \\ \equiv \prod_{j=1}^{m} \{v - g[\xi_j(x_2, \cdots, x_n, y, u), x_2, \cdots, x_n, y]\}. \end{aligned} \tag{6.4}$$

Then F is easily seen to be single-valued and analytic where $\Delta \neq 0$ near (0, 0, 0, 0), and bounded in modulus. Hence, by a theorem of Kistler,† F is analytic in a neighborhood of (0, 0, 0, 0). We shall now show that F is independent of $x_2, \cdots, x_n$.

At any point $(x_2, \cdots, x_n, y, u)$ where $\Delta \neq 0$, for $s > 1$,

$$\begin{aligned} F_{x_s} = - \sum_{q=1}^{m} \Bigg[\left(g_{x_1} \cdot \frac{\partial \xi_q}{\partial x_s} + g_{x_s} \right) \\ \cdot \prod_{j \neq q} \{v - g[\xi_j(x_2, \cdots, x_n, y, u), x_2, \cdots, x_n, y]\} \Bigg], \end{aligned} \tag{6.5}$$

with ξ_q as the first argument of g_{x_1} and of g_{x_s}. Now $f_{x_1} \neq 0$ at any point $[\xi_q(x_2, \cdots, x_n, y, u), x_2, \cdots, x_n, y]$, for since (6.3), solution of $H=0$, is satisfied, we see from (6.1) that if $f_{x_1}=0$ then $H_{x_1}=0$, and as both H and H_{x_1} would thus be zero, Δ would $=0$, contrary to hypothesis. Since $f_{x_1} \neq 0$, and (6.3), solution of $H=0$, is by (6.1) also a solution of $f(x, y) - u = 0$, it follows that

$$\frac{\partial \xi_q}{\partial x_s} = - f_{x_s}/f_{x_1},$$

with $x_1 = \xi_q$ in evaluating the right hand member. Substituting this value in (6.5), $q = 1, \cdots, m$, and using the hypothesis about the rank of the matrix of partial derivatives, we see from (6.5) that $F_{x_s}=0$ where $\Delta \neq 0$. By continuity, $F_{x_s} \equiv 0$. Hence F is independent of the x's, and we can write

$$F(x_2, \cdots, x_n, y, u, v) \equiv G(y, u, v). \tag{6.6}$$

Now if $u = f(x, y)$ and $v = g(x, y)$ for (x, y) near (0, 0) so that (u, v) is near (0, 0), then from (5.1) we see that $H(x, y, u) = 0$, so that x_1 must be one of the roots ξ_j of that equation given in (6.3). Hence, in one of the m factors

* Cf. Osgood II, chapter 2, §9; chapter 1, §6.

† Osgood II, chapter 3, §5, Theorem 1.

on the right hand side of (6.4), the first argument of g is x_1. Since $v=g(x, y)$, that factor is zero, so that $F=0$, and from (6.6) we then see that $G=0$. Thus $G[y, f(x, y), g(x, y)] \equiv 0$, as was to be proved.

Conversely, if $G(y, u, v)=0$ at a point near $(0, 0, 0)$, if $n>1$ and we take any $(x_2, \cdots, x_n)$ near $(0, \cdots, 0)$, then we see from (6.6) that at least one factor on the right hand side of (6.4) is zero. Hence if we let x_1 be a proper one of the roots (6.3) of $H=0$, then v will equal $g(x_1, x_2, \cdots, x_n, y)$, and, since $H(x, y, u)=0$, from (6.1) we see that $u=f(x_1, \cdots, x_n, y)$. Hence the second conclusion of the theorem is true when $n>1$. If $n=1$ there are no variables $x_2, \cdots, x_n$, and the argument simplifies, giving the stated result. Hence Theorem 6.I is true.

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